FIRST EIGENVALUE OF THE p-LAPLACE OPERATOR ALONG THE RICCI FLOW

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Abstract. In this paper, we mainly investigate continuity, monotonicity and differentiability for the first eigenvalue of the p-Laplace operator along the Ricci flow on closed manifolds. We show that the first p-eigenvalue is strictly increasing and differentiable almost everywhere along the Ricci flow under some curvature assumptions. In particular, for an orientable closed surface, we construct various monotonic quantities and prove that the first p-eigenvalue is differentiable almost everywhere along the Ricci flow without any curvature assumption, and therefore derive a p-eigenvalue comparison-type theorem when its Euler characteristic is negative.

1. Introduction

Given a compact Riemannian manifold (M^n, g_0) without boundary, the Ricci flow is the following evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

with the initial condition $g(x,0) = g_0(x)$, where R_{ij} denotes the Ricci tensor of the metric $g(t)$. The normalized Ricci flow is

$$\frac{\partial}{\partial \tilde{t}} \tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{2}{n}\tilde{r}\tilde{g}_{ij}, \tag{1.2}$$

where $\tilde{g}(\tilde{t}) := c(t)g(t)$, $\tilde{t}(t) := \int_0^t c(\tau)d\tau$ and

$$c(t) := \exp\left(\frac{2}{n}\int_0^t r(\tau)d\tau\right), \qquad \tilde{r} := \int_M \tilde{R}d\tilde{\mu} \Big/ \int_M d\tilde{\mu}, \tag{1.3}$$

($d\tilde{\mu}$ and $\tilde{R}$ denote the volume form and the scalar curvature of the metric $\tilde{g}(\tilde{t})$, respectively.) which preserves the volume of the initial manifold. Both evolution equations were introduced by R.S. Hamilton to approach the geometrization conjecture in [11]. Recently, studying the eigenvalues of geometric operator is a very powerful tool for understanding of Riemannian manifolds. In [23], G. Perelman introduced the functional

$$\mathcal{F}(g(t), f(t)) := \int_M \left(R + |\nabla f|^2\right) e^{-f} d\mu$$

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and showed that this functional is nondecreasing along the Ricci flow coupled to a backward heat-type equation. More precisely, if $g(t)$ is a solution to the Ricci flow (1.1) and the coupled $f(x,t)$ satisfies the following evolution equation:

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R,$$

then we have

$$\frac{\partial \mathcal{F}}{\partial t} = 2\int_M \left|Ric + \nabla^2 f\right|^2 e^{-f} d\mu.$$

If we define

$$\lambda(g(t)) := \inf_{f\neq 0}\left\{\mathcal{F}(g(t), f(t)) : f \in C^\infty(M), \int_M e^{-f} d\mu = 1\right\},$$

then $\lambda(g(t))$ is the lowest eigenvalue of the operator $-4\Delta + R$, and the increasing of the functional $\mathcal{F}(g,f)$ implies the increasing of $\lambda(g(t))$.

Later in [1], X.-D. Cao studied the eigenvalues λ and eigenfunctions f of the new operator $-\Delta + R/2$ satisfying $\int_M f^2 d\mu = 1$ on closed manifolds with nonnegative curvature operator. In fact he introduced

$$\lambda(f,t) := \int_M \left(-\Delta f + \frac{R}{2} f\right) f d\mu, \tag{1.4}$$

where f is a smooth function satisfying $\int_M f^2 d\mu = 1$ and obtained the following

Theorem A. (X.-D. Cao [1]) *On a closed Riemannian manifold with nonnegative curvature operator, the eigenvalues of the operator* $-\Delta + \frac{R}{2}$ *are nondecreasing under the unnormalized Ricci flow, i.e.*

$$\frac{d}{dt}\lambda(f,t) = 2\int_M Ric(\nabla f, \nabla f) + \int_M |Ric|^2 f^2 d\mu \geq 0. \tag{1.5}$$

In (1.5), when $\frac{d}{dt}\lambda(f,t)$ is evaluated at time t, f is the corresponding eigenfunction of $\lambda(t)$. Hence $\lambda(t)$ is nondecreasing.

Shortly thereafter J.-F. Li in [17] dropped the curvature assumption and also obtained the above result for the operator $-\Delta + \frac{R}{2}$. In fact, he used new entropy functionals to derive a general result.

Theorem B. (J.-F. Li [17]) *On a compact Riemannian manifold* $(M, g(t))$*, where* $g(t)$ *satisfies the unnormalized Ricci flow for* $t \in [0,T)$*, the lowest eigenvalue* λ_k *of the operator* $-4\Delta + kR$ $(k > 1)$ *is nondecreasing under the unnormalized Ricci flow. The monotonicity is strict unless the metric is Ricci-flat.*

At around the same time, X.-D. Cao in [2] also considered the general operator $-\Delta + cR$ $(c \geq 1/4)$, and derived the following exact monotonicity formula.

Theorem C. (X.-D. Cao [2]) *Let* $(M^n, g(t))$*,* $t \in [0,T)$*, be a solution of the unnormalized Ricci flow (1.1) on a closed manifold* M^n*. Assume that* $\lambda(t)$ *is the lowest eigenvalue of* $-\Delta + cR$ $(c \geq 1/4)$ *and* $f = f(x,t) > 0$ *satisfies*

$$-\Delta f(x,t) + cRf(x,t) = \lambda(t) f(x,t)$$

with $\int_M f^2 d\mu = 1$*. Then under the unnormalized Ricci flow, we have*

$$\frac{d}{dt}\lambda(t) = \frac{1}{2}\int_M |Ric + \nabla^2\varphi|^2 e^{-\varphi} d\mu + \frac{4c-1}{2}\int_M |Ric|^2 e^{-\varphi} d\mu \geq 0, \tag{1.6}$$

where $e^{-\varphi} = f^2$*.*

On the other hand, L. Ma in [20] considered the eigenvalues of the Laplace operator along the Ricci flow and proved the following result.

Theorem D. (L. Ma [20]) *Let $g = g(t)$ be the evolving metric along the unnormalized Ricci flow with $g(0) = g_0$ being the initial metric in M. Let D be a smooth bounded domain in (M, g_0). Let $\lambda > 0$ be the first eigenvalue of the Laplace operator of the metric $g(t)$. If there is a constant such that the scalar curvature $R \geq 2a$ in $D \times \{t\}$ and the Einstein tensor*

$$E_{ij} \geq -a g_{ij} \qquad \text{in} \quad D \times \{t\},$$

then we have $\lambda' \geq 0$, that is, λ is nondecreasing in t, furthermore, $\lambda'(t) > 0$ for the scalar curvature R not being the constant $2a$. The same monotonicity result is also true for other eigenvalues.

Moreover S.-C. Chang and P. Lu in [4] studied the evolution of Yamabe constant under the Ricci flow and gave a simple application. Motivated by the above works, in this paper we will study the first eigenvalue of the p-Laplace operator whose metric satisfying the Ricci flow. For the p-Laplace operator, besides many interesting properties between the eigenvalues of the p-Laplace operator and geometrical invariants were pointed out in fixed metrics (e.g. [10], [14], [16], [21]), the first author in [28] studied the monotonicity for the first eigenvalue of the p-Laplace operator along the Ricci flow on closed manifolds.

In this paper, on one hand we will improve those results in [28] and discuss the differentiability for the first eigenvalue of the p-Laplace operator along the unnormalized Ricci flow. Meanwhile we construct some monotonic quantities along the unnormalized Ricci flow. On the other hand, we will deal with the case of the normalized Ricci flow in the same way and give an interesting application. For the unnormalized Ricci flow, we first have

Theorem 1.1. *Let $g(t)$, $t \in [0, T)$, be a solution of the unnormalized Ricci flow (1.1) on a closed manifold M^n and $\lambda_{1,p}(t)$ be the first eigenvalue of the p-Laplace operator ($p > 1$) of $g(t)$. If there exists a nonnegative constant ϵ such that*

$$R_{ij} - \frac{R}{p} g_{ij} \geq -\epsilon g_{ij} \qquad \text{in} \quad M^n \times [0, T) \tag{1.7}$$

and

$$R \geq p \cdot \epsilon \quad \text{and} \quad R \not\equiv p \cdot \epsilon \qquad \text{in} \quad M^n \times \{0\}, \tag{1.8}$$

then $\lambda_{1,p}(t)$ is strictly increasing and differentiable almost everywhere along the unnormalized Ricci flow on $[0, T)$.

Remark 1.2. (1). In [28], the first author proved a similar result as in Theorem 1.1, where he assumed $p \geq 2$, inequality (1.7) and $R > p \cdot \epsilon$ in $M^n \times \{0\}$, which are a little stronger than assumptions of Theorem 1.1. The key difference is that the proof approach here is different from that in [28].

(2). As mentioned Remark 1.2 in [28], the time interval $[0, T)$ of Theorem 1.1 here may be not the maximal time interval of existence of the unnormalized Ricci flow. In fact if we trace (1.7) and assume that $p < n$, then we have an upper bound estimate for the scalar curvature ($\epsilon \neq 0$). But as we all known, curvature operator must be blow-up as $t \to T$ ($T < \infty$) when the curvature operator is positive and $[0, T)$ is the maximal time interval (see Theorem 14.1 in [11]).

(3). Theorem 1.1 still holds if the conditions (1.7) and (1.8) are replaced by $R_{ij} - \frac{R}{p} g_{ij} > -\epsilon g_{ij}$ in $M^n \times [0, T)$ and $R \geq p \cdot \epsilon$ in $M^n \times \{0\}$.

(4). For any closed 2-surface and 3-manifold, we can relax the above assumptions (1.7) and (1.8) to the only initial curvature assumptions by the Hamilton's maximum principle. We refer the reader to [28] for similar results.

Remark 1.3. Most recently, in [3] X.-D. Cao, S.-B. Hou and J. Ling derived a monotonicity formula for the first eigenvalue of $-\Delta + aR$ $(0 < a \leq 1/2)$ on closed surfaces with nonnegative scalar curvature under the Ricci flow. Meanwhile they obtained various monotonicity formulae and estimates for the first eigenvalue on closed surfaces.

Furthermore, if less curvature assumptions are given, we can construct two classes of monotonic (increasing and decreasing) quantities about the first eigenvalue of the p-Laplace operator along the unnormalized Ricci flow. We refer the reader to Section 4 for the more detailed discussions (see Theorems 4.3 and 4.5, and Corollary 4.6).

For the normalized Ricci flow, unfortunately we may not get any monotonicity for the first eigenvalue of the p-Laplace operator in general. However, if we know the first p-eigenvalue differentiability along the unnormalized Ricci flow, from the relation to the unnormalized Ricci flow, we can give another way to derive the first p-eigenvalue differentiability along the normalized Ricci flow (see Theorem 5.1 of Section 5).

Besides, the most important result is that we can construct various monotonic quantities about the first eigenvalue of the p-Laplace operator along the normalized Ricci flow on closed 2-surfaces without any curvature assumption. This also leads to the first p-eigenvalue differentiability along the normalized Ricci flow on closed 2-surfaces without any curvature assumption.

Theorem 1.4. *Let $\tilde{g}(\tilde{t})$, $\tilde{t} \in [0,\infty)$, be a solution of the normalized Ricci flow (1.2) on a closed surface M^2 and let $\lambda_{1,p}(\tilde{t})$ be the first eigenvalue of the p-Laplace operator of the metric $\tilde{g}(\tilde{t})$. Then each of the following quantities*

(1) $\lambda_{1,p}(\tilde{t}) \cdot \left(\frac{\rho_0}{\tilde{r}} - \frac{\rho_0}{\tilde{r}} e^{\tilde{r}\tilde{t}} + e^{\tilde{r}\tilde{t}}\right)^{p/2}$ $\quad (p \geq 2)$,
$\lambda_{1,p}(\tilde{t}) \cdot \left(\frac{\rho_0}{\tilde{r}} - \frac{\rho_0}{\tilde{r}} e^{\tilde{r}\tilde{t}} + e^{\tilde{r}\tilde{t}}\right) \cdot \exp\left[\left(1 - \frac{p}{2}\right)\frac{C}{\tilde{r}} e^{\tilde{r}\tilde{t}}\right]$ $\quad (1 < p < 2)$, $\quad$ if $\chi(M^2) < 0$;

(2) $\lambda_{1,p}(\tilde{t}) \cdot \left(1 + C\tilde{t}\right)^{p/2}$ $\quad (p \geq 2)$,
$\lambda_{1,p}(\tilde{t}) \cdot \left(1 + C\tilde{t}\right) \cdot e^{(1-p/2)C\tilde{t}}$ $\quad (1 < p < 2)$, $\quad$ if $\chi(M^2) = 0$;

(3) $\ln \lambda_{1,p}(\tilde{t}) + \frac{p}{2} \cdot \left(\frac{C}{\tilde{r}} e^{\tilde{r}\tilde{t}} + \tilde{r}\tilde{t}\right)$ $\quad (p \geq 2)$,
$\ln \lambda_{1,p}(\tilde{t}) + \left(2 - \frac{p}{2}\right) \frac{C}{\tilde{r}} e^{\tilde{r}\tilde{t}} + \tilde{r}\tilde{t}$ $\quad (1 < p < 2)$, $\quad$ if $\chi(M^2) > 0$

is increasing and therefore $\lambda_{1,p}(\tilde{t})$ is differentiable almost everywhere along the normalized Ricci flow on $[0,\infty)$, where $\chi(M^2)$ denotes its Euler characteristic, $\rho_0 := \inf_{M^2} R(0)$ and $C > 0$ is a constant depending only on the initial metric.

In the same way, we can also obtain the decreasing quantities on closed 2-surfaces.

Theorem 1.5. *Under the same assumptions as in Theorem 1.4, then each of the following quantities*

(1) $\ln \lambda_{1,p}(\tilde{t}) - \frac{p}{2} \cdot \frac{C}{\tilde{r}} e^{\tilde{r}\tilde{t}}$ $\quad (p \geq 2)$,
$\lambda_{1,p}(\tilde{t}) \cdot \left(\frac{\rho_0}{\tilde{r}} - \frac{\rho_0}{\tilde{r}} e^{\tilde{r}\tilde{t}} + e^{\tilde{r}\tilde{t}}\right)^{\left(\frac{p}{2}-1\right)} \cdot \exp\left(-\frac{C}{\tilde{r}} e^{\tilde{r}\tilde{t}}\right)$ $\quad (1 < p < 2)$, $\quad$ if $\chi(M^2) < 0$;

(2) $\ln\lambda_{1,p}(\tilde t)-\frac{p}{2}\cdot C\tilde t$ $(p\geq 2)$,
$\lambda_{1,p}(\tilde t)\cdot\left(1+C\tilde t\right)^{\left(\frac{p}{2}-1\right)}\cdot e^{-C\tilde t}$ $(1<p<2)$ if $\chi(M^2)=0$;
(3) $\ln\lambda_{1,p}(\tilde t)-\frac{p}{2}\cdot\frac{C}{\tilde r}e^{\tilde r\tilde t}$ $(p\geq 2)$,
$\ln\lambda_{1,p}(\tilde t)-\left(2-\frac{p}{2}\right)\frac{C}{\tilde r}\cdot e^{\tilde r\tilde t}-\left(1-\frac{p}{2}\right)\tilde r\tilde t$ $(1<p<2)$ if $\chi(M^2)>0$

is decreasing and therefore $\lambda_{1,p}(\tilde t)$ is differentiable almost everywhere along the normalized Ricci flow on $[0,\infty)$, where $\chi(M^2)$, ρ_0 and C are as in Theorem 1.4.

Remark 1.6. We may apply similar techniques above to obtain interesting monotonic quantities about the first eigenvalue of the p-Laplace operator along the normalized Ricci flow in high-dimensional cases under some curvature assumptions, but the proof needs more computing. Here we omit this aspect.

Some parts of results for $p=2$ above were proved by L. Ma [20] and J. Ling [19]. But our method of proof is different from theirs. Their proofs strongly depend on the differentiability for the eigenvalues and the corresponding eigenfunctions. But in our setting $(p\geq 2)$ it is not clear whether the eigenvalue or the corresponding eigenfunction is differentiable in advance. Our method is similar to X.-D. Cao's trick in [1], which does not depend on the differentiability for the eigenvalues or the corresponding eigenfunctions.

With the help of Theorem 1.4, our below topic is to extend an earlier J. Ling's result for $p=2$ (see [18]). Here we call it p-eigenvalue comparison-type theorem. For the convenience of introducing our result, we shall state a well-known fact, which was proved by R.S. Hamilton and B. Chow (see also [7], chapter 5 for details).

Theorem E. (Chow-Hamilton, [5] and [12]) *If (M^2,g) is a closed surface, there exists a unique solution $g(t)$ of the normalized Ricci flow (1.2). The solution exists for all the time. As $t\to\infty$, the metrics $g(t)$ converge uniformly in any C^k-norm to a smooth metric $\bar g(=g(\infty))$ of constant curvature.*

Let (M^2,g) be a closed surface. Let K_g, κ_g, $\mathrm{Area}_g(M^2)$ denote the Gauss curvature, the minimum of the Gauss curvature, the area of the surface M^2, respectively. $\lambda_{1,p}(g)$ denotes the first eigenvalue of the p-Laplace operator with respect to the metric g. Then we prove that

Theorem 1.7. *(p-eigenvalue comparison-type theorem). Suppose that (M^2,g) is a closed surface with its Euler characteristic $\chi(M^2)<0$. The Ricci flow with initial metric g converges uniformly to a smooth metric $\bar g$ of constant curvature. Then for any $p\geq 2$,*

$$\frac{\lambda_{1,p}(\bar g)}{\lambda_{1,p}(g)}\geq\left(\frac{\kappa_{\bar g}}{\kappa_g}\right)^{p/2}\tag{1.9}$$

and the constant Gauss curvature for metric $\bar g$ is $\kappa_{\bar g}=2\pi\chi(M^2)/\mathrm{Area}_g(M^2)$.

In conclusion, our new contribution of this paper is to obtain the monotonicity for the first eigenvalue of the p-Laplace operator, and construct many monotonic quantities involving the first eigenvalue of the p-Laplace operator along the Ricci flow under some different curvature assumptions. By the monotonic property, we can judge the differentiability in some sense for the first eigenvalue of a nonlinear operator with respect to evolving metrics. Using the same idea of our arguments,

we easily see that Perelman's eigenvalue is differentiable almost everywhere[1]. From Theorem 1.4 above and Corollary 5.4 below, we also see that the first eigenvalue of the p-Laplace operator is differentiable almost everywhere along the Ricci flow on closed 2-surfaces without any curvature assumption. For high-dimensional case, the similar differentiability property still holds as long as some curvature conditions are satisfied. Of course, the proofs of these results involve many skilled arguments and computations. Finally, it should be remarked that it is still an open question whether its corresponding eigenfunction is differentiable with respect to t-variable along the Ricci flow.

The rest of this paper is organized as follows. In Section 2, we will recall some notations about p-Laplace, and prove that $\lambda_{1,p}(g(t))$ is a continuous function along the Ricci flow. In Section 3, we will give Proposition 3.1. Using this proposition, we can finish the proof of Theorem 1.1. In Section 4, we will construct two classes of monotonic quantities about the first eigenvalue of the p-Laplace operator along the unnormalized Ricci flow. In Section 5, we will discuss the normalized Ricci flow case and mainly prove Theorems 1.4 and 1.5. In Section 6, we shall prove p-eigenvalue comparison-type theorem, i.e., Theorem 1.7. In Section 7, we will use the same method to study the first eigenvalue of the p-Laplace with respect to general evolving metrics, especially to the Yamabe flow.

2. Preliminaries

In this section, we will first recall some definitions about the p-Laplace operator and give the definition for the first eigenvalue of the p-Laplace operator under the Ricci flow on a closed manifold. Then we will show that the first eigenvalue of the p-Laplace operator is a continuous function along the Ricci flow.

Let M^n be an n-dimensional connected closed Riemannian manifold and $g(t)$ be a smooth solution of the Ricci flow on the time interval $[0, T)$. Consider the nonzero first eigenvalue of the p-Laplace operator ($p > 1$) at time t (also called the first p-eigenvalue), where $0 \leq t < T$, i.e.,

$$\lambda_{1,p}(t) := \inf_{f \neq 0} \left\{ \frac{\int_M |df|^p d\mu}{\int_M |f|^p d\mu} : f \in W^{1,p}(M), \quad \int_M |f|^{p-2} f d\mu = 0 \right\}. \tag{2.1}$$

Obviously, this infimum does not change when $W^{1,p}(M)$ is replaced by $C^\infty(M)$. For the fixed time, this infimum is achieved by a $C^{1,\alpha}$ ($0 < \alpha < 1$) eigenfunction f_p (see [25] and [26]). The corresponding eigenfunction f_p satisfies the following Euler-Lagrange equation

$$\Delta_p f_p = -\lambda_{1,p}(t) |f_p|^{p-2} f_p, \tag{2.2}$$

where Δ_p ($p > 1$) is the p-Laplace operator with respect to $g(t)$, given by

$$\Delta_{p_{g(t)}} f := \mathrm{div}_{g(t)} \left(|df|^{p-2}_{g(t)} df \right). \tag{2.3}$$

If $p = 2$, the p-Laplace operator reduces to the Laplace-Beltrami operator. The most difference between two operators is that the p-Laplace operator is a nonlinear operator in general, but the Laplace-Beltrami operator is a linear operator.

Note that it is not clear whether the first eigenvalue of the p-Laplace operator or its corresponding eigenfunction is C^1-differentiable along the Ricci flow. When

[1]Note that many literatures have pointed out that the differentiability for Perelman's eigenvalue follows from eigenvalue perturbation theory (see also Section 2).

$p = 2$, where Δ_p is the Laplace-Beltrami operator, many papers have pointed out that their differentiability follows from eigenvalue perturbation theory (for example, see [2], [13], [15] and [24]). But $p \neq 2$, as far as we are aware, the differentiability for the first eigenvalue of the p-Laplace operator or its corresponding eigenfunction along the Ricci flow has not been known until now. Even we have not known whether they are locally Lipschitz. So we can not use the method used by L. Ma to derive the monotonicity for the first eigenvalue of the p-Laplace operator.

Although we do not know the differentiability for $\lambda_{1,p}(t)$, we will see that $\lambda_{1,p}(g(t))$ in fact is a continuous function along the Ricci flow on $[0, T)$. This is a consequence of the following elementary result.

Theorem 2.1. *If g_1 and g_2 are two metrics which satisfy*

$$(1+\varepsilon)^{-1} g_1 \leq g_2 \leq (1+\varepsilon) g_1,$$

then for any $p > 1$, we have

$$(1+\varepsilon)^{-(n+\frac{p}{2})} \leq \frac{\lambda_{1,p}(g_1)}{\lambda_{1,p}(g_2)} \leq (1+\varepsilon)^{(n+\frac{p}{2})}. \tag{2.4}$$

In particular, $\lambda_{1,p}(g(t))$ is a continuous function in the t-variable.

To prove this theorem, we first need the following fact. Let (M^n, g) be an n-dimensional closed Riemannian manifold. For any non-constant function f, consider the following C^1-function on $s \in (-\infty, \infty)$

$$F(s) := \int_{M^n} |f + s|^p \, d\mu_g, \ \ (p > 1).$$

Lemma 2.2. *There exists a unique $s_0 \in (-\infty, \infty)$ such that*

$$F(s_0) = \min_{s \in \mathbb{R}} F(s) \quad \text{if and only if} \quad \int_M |f + s_0|^{p-2} (f + s_0) d\mu_g = 0. \tag{2.5}$$

Proof. Note that the function $|x|^p$ $(p > 1)$ is a strictly convex function on $x \in \mathbb{R}$. Meanwhile we can also check that

$$\lim_{|s| \to +\infty} F(s) \to +\infty, \qquad F'(s) = p \int_M |f + s|^{p-2} (f + s) \, d\mu_g.$$

Therefore $F(s)$ is a strictly convex function and there exists a unique $s_0 \in (-\infty, +\infty)$ such that

$$F(s_0) = \min_{s \in \mathbb{R}} F(s) \quad \text{and} \quad F'(s) = p \int_M |f + s_0|^{p-2} (f + s_0) d\mu_g = 0. \tag{2.6}$$

□

Now using Lemma 2.2, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Since the volume form $d\mu$ has degree $n/2$ in g, we have

$$(1+\varepsilon)^{-n/2} d\mu_{g_1} \leq d\mu_{g_2} \leq (1+\varepsilon)^{n/2} d\mu_{g_1}. \tag{2.7}$$

Taking f be the first eigenfunction of Δ_p with respect to the metric g_1, we see that

$$\lambda_{1,p}(g_1) = \frac{\int_M |df|_{g_1}^p d\mu_{g_1}}{\int_M |f|^p d\mu_{g_1}} \quad \text{and} \quad \int_M |f|^{p-2} f d\mu_{g_1} = 0. \tag{2.8}$$

Since $\int_M |f|^{p-2} f d\mu_{g_1} = 0$, Lemma 2.2 implies

$$\int_M |f|^p d\mu_{g_1} = \min_{s\in\mathbb{R}} \int_M |f+s|^p \, d\mu_{g_1}.$$

Hence by (2.8), we conclude that

$$\lambda_{1,p}(g_1) = \frac{\int_M |df|^p_{g_1} d\mu_{g_1}}{\int_M |f|^p d\mu_{g_1}} \geq \frac{\int_M |d(f+s)|^p_{g_1} d\mu_{g_1}}{\int_M |f+s|^p d\mu_{g_1}}. \tag{2.9}$$

Keep in mind that under another metric g_2, for function $F(s) = \int_M |f+s|^p \, d\mu_{g_2}$, there exists a unique $s_0 \in (-\infty, +\infty)$ such that

$$F(s_0) = \min_{s\in\mathbb{R}} F(t) \quad \text{and} \quad F'(s) = p\int_M |f+s_0|^{p-2} (f+s_0) d\mu_{g_2} = 0. \tag{2.10}$$

Using (2.7), from (2.9) we conclude that

$$\lambda_{1,p}(g_1) \geq \frac{\int_M |d(f+s)|^p_{g_1} d\mu_{g_1}}{\int_M |f+s|^p d\mu_{g_1}} \geq (1+\varepsilon)^{-(n+\frac{p}{2})} \cdot \frac{\int_M |d(f+s)|^p_{g_2} d\mu_{g_2}}{\int_M |f+s|^p d\mu_{g_2}}. \tag{2.11}$$

Letting $s = s_0$ in (2.11) yields

$$\lambda_{1,p}(g_1) \geq (1+\varepsilon)^{-(n+\frac{p}{2})} \cdot \frac{\int_M |d(f+s_0)|^p_{g_2} d\mu_{g_2}}{\int_M |f+s_0|^p d\mu_{g_2}} \geq (1+\varepsilon)^{-(n+\frac{p}{2})} \cdot \lambda_{1,p}(g_2), \tag{2.12}$$

where for the last inequality we used $\int_M |f+s_0|^{p-2} (f+s_0) d\mu_{g_2} = 0$ and the definition for the first p-eigenvalue with respect to the metric g_2.

From the course of this proof, we easily see that (2.12) still holds if we exchange g_1 and g_2. Hence

$$(1+\varepsilon)^{-(n+\frac{p}{2})} \leq \frac{\lambda_{1,p}(g_1)}{\lambda_{1,p}(g_2)} \leq (1+\varepsilon)^{(n+\frac{p}{2})}. \tag{2.13}$$

This completes the proof of Theorem 2.1. □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 in introduction. In order to achieve this, we first prove the following proposition. Our proof involves choosing a proper smooth function, which seems to be a delicate trick.

Proposition 3.1. *Let $g(t)$, $t \in [0,T)$, be a solution of the unnormalized Ricci flow (1.1) on a closed manifold M^n and let $\lambda_{1,p}(t)$ be the first eigenvalue of the p-Laplace operator along this flow. For any $t_1, t_2 \in [0,T)$ and $t_2 \geq t_1$, we have*

$$\lambda_{1,p}(t_2) \geq \lambda_{1,p}(t_1) + \int_{t_1}^{t_2} \mathcal{G}(g(\xi), f(\xi)) d\xi, \tag{3.1}$$

where

$$\mathcal{G}(g(t), f(t)) := p\int_M |df|^{p-2} Ric(\nabla f, \nabla f) d\mu - p\int_M \Delta_p f \frac{\partial f}{\partial t} d\mu - \int_M |df|^p R d\mu \tag{3.2}$$

and where $f(t)$ is any C^∞ function satisfying $\int_M |f|^p d\mu = 1$ and $\int_M |f|^{p-2} f d\mu = 0$, such that at time t_2, $f(t_2)$ is the corresponding eigenfunction of $\lambda_{1,p}(t_2)$.

Proof. Set

$$G(g(t),f(t)):=\int_M |df(t)|^p_{g(t)}d\mu_{g(t)}.$$

We *claim* that, for any time $t_2\in(0,T)$, there exists a C^∞ function $f(t)$ satisfying

$$\int_M |f(t)|^p d\mu_{g(t)}=1 \qquad \text{and} \qquad \int_M |f(t)|^{p-2}f(t)d\mu_{g(t)}=0 \tag{3.3}$$

and such that at time t_2, $f(t_2)$ is the eigenfunction for $\lambda_{1,p}(t_2)$ of $\Delta_{p_{g(t_2)}}$. To see this, at time t_2, we first let $f_2=f(t_2)$ be the eigenfunction for the eigenvalue $\lambda_{1,p}(t_2)$ of $\Delta_{p_{g(t_2)}}$. Then we consider the following smooth function

$$h(t)=f_2\left[\frac{\det(g_{ij}(t_2))}{\det(g_{ij}(t))}\right]^{\frac{1}{2(p-1)}} \tag{3.4}$$

under the Ricci flow $g_{ij}(t)$. Later we normalize this smooth function

$$f(t)=\frac{h(t)}{\left(\int_M |h(t)|^p d\mu\right)^{1/p}} \tag{3.5}$$

under the Ricci flow $g_{ij}(t)$. From above, we can easily check that $f(t)$ satisfies (3.3).

By the definition for $\lambda_{1,p}(t_2)$, we have

$$\lambda_{1,p}(t_2)=G(g(t_2),f(t_2)). \tag{3.6}$$

Notice that under the unnormalized Ricci flow,

$$\frac{\partial}{\partial t}|df|^p=p|df|^{p-2}\left(R_{ij}f_if_j+f_i\frac{\partial f_i}{\partial t}\right), \qquad \frac{\partial}{\partial t}(d\mu)=-Rd\mu, \tag{3.7}$$

where f_i and R_{ij} denote the covariant derivative of f and Ricci curvature with respect to the Levi-Civita connection of $g(t)$, respectively.

Note that $G(g(t),f(t))$ is a smooth function with respect to t-variable. So

$$\begin{aligned}
\mathcal{G}(g(t),f(t)):&=\frac{d}{dt}G(g(t),f(t))\\
&=\int_M\frac{\partial}{\partial t}|df|^p d\mu-\int_M|df|^pRd\mu\\
&=p\int_M|df|^{p-2}R_{ij}f_if_jd\mu+p\int_M|df|^{p-2}f_i\frac{\partial}{\partial t}(f_i)d\mu-\int_M|df|^pRd\mu\\
&=p\int_M|df|^{p-2}R_{ij}f_if_jd\mu-p\int_M\nabla_i(|df|^{p-2}f_i)\frac{\partial f}{\partial t}d\mu-\int_M|df|^pRd\mu\\
&=p\int_M|df|^{p-2}R_{ij}f_if_jd\mu-p\int_M\Delta_pf\frac{\partial f}{\partial t}d\mu-\int_M|df|^pRd\mu,
\end{aligned} \tag{3.8}$$

where we used (3.7). Taking integration on the both sides of (3.8) between t_1 and t_2, we conclude that

$$G(g(t_2),f(t_2))-G(g(t_1),f(t_1))=\int_{t_1}^{t_2}\mathcal{G}(g(\xi),f(\xi))d\xi, \tag{3.9}$$

where $t_1\in[0,T)$ and $t_2\geq t_1$. Noticing $G(g(t_1),f(t_1))\geq\lambda_{1,p}(t_1)$ and combining (3.6) with (3.9), we arrive at

$$\lambda_{1,p}(t_2)\geq\lambda_{1,p}(t_1)+\int_{t_1}^{t_2}\mathcal{G}(g(\xi),f(\xi))d\xi,$$

where $\mathcal{G}(g(\xi), f(\xi))$ satisfies (3.8). □

In the following of this section, we will finish the proof of Theorem 1.1 using Proposition 3.1.

Proof of Theorem 1.1. In fact, we only need to show that $\mathcal{G}(g(t), f(t)) > 0$ in Proposition 3.1. Notice that at time t_2, $\lambda_{1,p}(t_2)$ is the first eigenvalue and $f(t_2)$ is the corresponding eigenfunction. Therefore at time t_2, we have

(3.10)
$$\begin{aligned}\mathcal{G}(g(t_2), f(t_2)) &= p\int_M |df|^{p-2}R_{ij}f_if_jd\mu - p\int_M \Delta_p f\frac{\partial f}{\partial t}d\mu - \int_M |df|^pRd\mu\\ &= p\int_M |df|^{p-2}R_{ij}f_if_jd\mu + p\lambda_{1,p}(t_2)\int_M |f|^{p-2}f\frac{\partial f}{\partial t}d\mu\int_M |df|^pRd\mu,\end{aligned}$$

where we used $\Delta_p f(t_2) = -\lambda_{1,p}(t_2)|f(t_2)|^{p-2}f(t_2)$.

Under the unnormalized Ricci flow, from the constraint condition

$$\frac{d}{dt}\int_M |f(t)|^p\, d\mu_{g(t)} = 0,$$

we know that

$$p\int_M |f|^{p-2}f\frac{\partial f}{\partial t}d\mu = \int_M |f|^pRd\mu. \tag{3.11}$$

Substituting this into the above formula (3.10) and combining the assumption of Theorem 1.1: $R_{ij} - \frac{R}{p}g_{ij} \geq -\epsilon g_{ij}$ in $M^n \times [0, T)$, we obtain

(3.12)
$$\begin{aligned}\mathcal{G}(g(t_2), f(t_2)) &= p\int_M |df|^{p-2}R_{ij}f_if_jd\mu + p\lambda_{1,p}(t_2)\int_M |f|^{p-2}f\frac{\partial f}{\partial t}d\mu - \int_M |df|^pRd\mu\\ &= \lambda_{1,p}(t_2)\int_M |f|^pRd\mu + \int_M |df|^{p-2}(pR_{ij} - Rg_{ij})f_if_jd\mu\\ &\geq \lambda_{1,p}(t_2)\int_M |f|^pRd\mu - p\cdot\epsilon\int_M |df|^pd\mu\\ &= \lambda_{1,p}(t_2)\int_M |f|^p(R - p\cdot\epsilon)d\mu.\end{aligned}$$

Meanwhile we also have another assumption of Theorem 1.1 on the scalar curvature

$$R \geq p\cdot\epsilon \text{ and } R \not\equiv p\cdot\epsilon \qquad \text{in} \quad M^n \times \{0\}.$$

It is well-known that $R \geq p\cdot\epsilon$ is preserved by the unnormalized Ricci flow. Furthermore by the strong maximum principle (for example, see Proposition 12.47 of Chapter 12 in [8]), we conclude that

$$R > p\cdot\epsilon \qquad \text{in} \quad M^n \times [0, T). \tag{3.13}$$

Plugging this into (3.12) implies $\mathcal{G}(g(t_2), f(t_2)) > 0$. Notice that $f(x, t)$ is a smooth function with respect to t-variable. Therefore we can arrive at $\mathcal{G}(g(\xi), f(\xi)) > 0$ in any sufficient small neighborhood of t_2. Hence

$$\int_{t_1}^{t_2} \mathcal{G}(g(\xi), f(\xi))d\xi > 0 \tag{3.14}$$

for any $t_1 < t_2$ sufficiently close to t_2. In the end, by Proposition 3.1, we conclude

$$\lambda_{1,p}(t_2) > \lambda_{1,p}(t_1)$$

for any $t_1 < t_2$ sufficiently close to t_2. Since $t_2 \in [0, T)$ is arbitrary, then the first part of Theorem 1.1 follows.

As for the differentiability for $\lambda_{1,p}(t)$, since $\lambda_{1,p}(t)$ is increasing on the time interval $[0, T)$ under curvature conditions of the theorem, by the classical Lebesgue's theorem (for example, see Chapter 4 in [22]), it is easy to see that $\lambda_{1,p}(t)$ is differentiable almost everywhere on $[0, T)$. □

Remark 3.2. (1). Our proof of the first p-eigenvalue monotonicity is not derived from the differentiability for $\lambda_{1,p}(t)$ or its corresponding eigenfunction. In fact we do not know whether they are differentiable in advance. It would be interesting to find out whether the corresponding eigenfunction of the p-Laplace operator is a C^1-differentiable function with respect to t-variable along the Ricci flow on a closed manifold M^n. If it is true, we can use L. Ma's method to get our result.

(2). If $p = 2$, the above theorem is similar to L. Ma's main result for the first eigenvalue of the Laplace operator in [20].

(3). Using this method, we can not get any monotonicity for higher order eigenvalues of the p-Laplace operator.

4. Monotonic quantities along unnormalized Ricci flow

Motivated by the works of X.-D. Cao [1] and [2], in this section, we first introduce a new smooth eigenvalue function (see (4.1) below), and then we give the following useful Lemma 4.1, resembling Proposition 3.1 of Section 3. Using this lemma, we can obtain two classes of interesting monotonic quantities along the unnormalized Ricci flow, that is, Theorem 4.3, Theorem 4.5 and Corollary 4.6. Then by means of those monotonic quantities, we can prove the differentiability for the first eigenvalue of the p-Laplace operator along the unnormalized Ricci flow.

Let M^n be an n-dimensional connected closed Riemannian manifold and $\tilde{g}(\tilde{t})$ be a smooth solution of the normalized Ricci flow on the time interval $[0, \infty)$. Now we can define a general smooth eigenvalue function

$$\lambda_{1,p}(\tilde{f}, \tilde{t}) := \int_M \tilde{\Delta}_{p_{\tilde{g}(\tilde{t})}} \tilde{f} \cdot \tilde{f} d\tilde{\mu} = \int_M |d\tilde{f}|^p d\tilde{\mu}, \tag{4.1}$$

where $\tilde{f}$ is a smooth function and satisfies the following equalities

$$\int_M |\tilde{f}(\tilde{t})|^p d\tilde{\mu}_{\tilde{g}(\tilde{t})} = 1 \qquad \text{and} \qquad \int_M |\tilde{f}(\tilde{t})|^{p-2} \tilde{f}(\tilde{t}) d\tilde{\mu}_{\tilde{g}(\tilde{t})} = 0. \tag{4.2}$$

From the proof of Proposition 3.1, we see that the above restriction (4.2) can be achieved.

Obviously, at time t_0, if $\tilde{f}$ is the corresponding eigenfunction of the first eigenvalue $\lambda_{1,p}(t_0)$, then

$$\lambda_{1,p}(\tilde{f}, t_0) = \lambda_{1,p}(t_0).$$

For the convenient of writing, we shall drop the tilde over all the variables used above to distinguish between the normalized and unnormalized Ricci flow.

Lemma 4.1. *If $\lambda_{1,p}(t)$ is the first eigenvalue of $\Delta_{p_{g(t)}}$, whose metric satisfying the normalized Ricci flow and $f(t_0)$ is the corresponding eigenfunction of $\lambda_{1,p}(t)$ at*

time t_0, then we have

$$\begin{aligned}\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} =&\lambda_{1,p}(f(t_0),t_0)\int_M |f|^p R d\mu + p\int_M |df|^{p-2}R_{ij}f_if_j d\mu\\ &-\int_M |df|^p R d\mu - \frac{p}{n}r\lambda_{1,p}(f(t_0),t_0).\end{aligned}\tag{4.3}$$

In particular, for any closed 2-surface, we have

$$\begin{aligned}\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} =& \lambda_{1,p}(f(t_0),t_0)\int_M |f|^p R d\mu + \left(\frac{p}{2}-1\right)\int_M |df|^p R d\mu\\ &-\frac{p}{2}r\lambda_{1,p}(f(t_0),t_0),\end{aligned}\tag{4.4}$$

where f evolves by (4.2) with the initial data $f(t_0)$.

Proof. The proof is by direct computations. Here we need to use

$$\frac{\partial}{\partial t}|df|^p = p|df|^{p-2}\left(R_{ij}f_if_j - \frac{r}{n}g_{ij}f_if_j + f_i\frac{\partial f_i}{\partial t}\right),\quad \frac{\partial}{\partial t}(d\mu) = (r-R)d\mu.$$

Then

$$\begin{aligned}\frac{d\lambda_{1,p}(f,t)}{dt}\Big|_{t=t_0} &= p\int_M |df|^{p-2}R_{ij}f_if_j d\mu + p\int_M |df|^{p-2}f_i\frac{\partial\,(f_i)}{\partial t}d\mu\\ &\quad - p\int_M |df|^p \frac{r}{n}d\mu + \int_M |df|^p(r-R)d\mu\\ &= p\int_M |df|^{p-2}R_{ij}f_if_j d\mu - p\int_M \nabla_i\left(|df|^{p-2}f_i\right)\frac{\partial f}{\partial t}d\mu\\ &\quad -\frac{p}{n}r\lambda_{1,p}(f(t_0),t_0) + \int_M |df|^p(r-R)d\mu\\ &= p\int_M |df|^{p-2}R_{ij}f_if_j d\mu + p\lambda_{1,p}(f(t_0),t_0)\int_M |f|^{p-2}f\frac{\partial f}{\partial t}d\mu\\ &\quad -\frac{p}{n}r\lambda_{1,p}(f(t_0),t_0) + \int_M |df|^p(r-R)d\mu,\end{aligned}\tag{4.5}$$

where we used f is the eigenfunction at time t_0, i.e., equation (2.2) at time t_0. Note that by (4.2), we have

$$p\int_M |f|^{p-2}f\frac{\partial f}{\partial t}d\mu = \int_M |f|^p(R-r)d\mu.\tag{4.6}$$

Plugging this into (4.5) yields the desired (4.3). For any closed 2-surface, we have $R_{ij} = \frac{R}{2}g_{ij}$. Hence (4.4) follows from (4.3). □

Remark 4.2. In [28], the first author used a similar method and proved a similar result for the unnormalized Ricci flow (see Proposition 2.1 in [28]).

In the following we first obtain increasing quantities along the unnormalized Ricci flow by using Lemma 4.1.

Theorem 4.3. *Let $g(t)$ and $\lambda_{1,p}(t)$ $(p>1)$ be the same as in Theorem 1.1. If $\rho_0 := \inf_M R(0) > 0$ and*

$$R_{ij} - \tfrac{R}{p}g_{ij}(t) > 0 \qquad \text{in} \quad M^n\times[0,T),\tag{4.7}$$

then the following quantity

$$\lambda_{1,p}(t) \cdot \left(\rho_0^{-1} - 2at\right)^{\frac{1}{2a}}, \tag{4.8}$$

is strictly increasing and therefore $\lambda_{1,p}(t)$ *is differentiable almost everywhere along the unnormalized Ricci flow on* $[0, T')$, *where* $a := \max\{\frac{1}{n}, \frac{n}{p^2}\}$ *and* $T' := \min\{\frac{1}{2a\rho_0}, T\}$.

Proof. We assume that at time $t_0 \in [0, T)$, if g is the corresponding eigenfunction of $\lambda_{1,p}(t_0)$, then under the unnormalized Ricci flow, we can construct a smooth function f satisfying

$$\int_M |f(t)|^p d\mu_{g(t)} = 1 \qquad \text{and} \qquad \int_M |f(t)|^{p-2} f(t) d\mu_{g(t)} = 0,$$

and such that at time $t = t_0$, $f = g$ is the eigenfunction of $\lambda_{1,p}(t_0)$. Meanwhile we can define a general smooth eigenvalue function $\lambda_{1,p}(f, t)$ as (4.1) under the unnormalized Ricci flow. Obviously, we have

$$\lambda_{1,p}(f(t_0), t_0) = \lambda_{1,p}(t_0).$$

According to (4.3) of Lemma 4.1, we have

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} = \lambda_{1,p}(f(t_0), t_0)\int_M |f|^p R d\mu + \int_M |df|^{p-2}(pR_{ij} - Rg_{ij}) f_i f_j d\mu, \tag{4.9}$$

where f is a smooth function satisfying the above assumptions. By the assumption $R_{ij} - \frac{R}{p} g_{ij} > 0$ of Theorem 4.3, we get

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} > \lambda_{1,p}(f(t_0), t_0)\int_M |f|^p R d\mu. \tag{4.10}$$

The evolution of the scalar curvature R under the unnormalized Ricci flow

$$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2$$

and inequality $|Ric|^2 \geq aR^2$ ($a := \max\{\frac{1}{n}, \frac{n}{p^2}\}$) imply

$$\frac{\partial}{\partial t} R \geq \Delta R + 2aR^2. \tag{4.11}$$

Since the solutions to the corresponding ODE

$$d\rho/dt = 2a\rho^2$$

are

$$\rho(t) = \frac{1}{{\rho_0}^{-1} - 2at}, \quad t \in [0, T'),$$

where $\rho_0 := \inf_M R(0)$ and $T' := \min\{(2a\rho_0)^{-1}, T\}$. Using the maximum principle to (4.11), we have $R(x,t) \geq \rho(t)$. Therefore (4.10) becomes

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} > \lambda_{1,p}(f(t_0), t_0) \cdot \rho(t_0).$$

Note that $\lambda_{1,p}(f,t)$ and $\rho(t)$ are both smooth functions with respect to t-variable. Hence we have

$$\frac{d}{dt}\lambda_{1,p}(f,t) > \lambda_{1,p}(f(t), t) \cdot \rho(t) \tag{4.12}$$

in any sufficiently small neighborhood of t_0. Now integrating the above inequality with respect to time t on time interval $[t_1, t_0]$, we get

$$\begin{aligned}&\ln\lambda_{1,p}(f(t_0),t_0) - \ln\lambda_{1,p}(f(t_1),t_1)\\ &> \left(-\frac{1}{2a}\right)\cdot\ln\left(\rho_0^{-1}-2at\right)\Big|_{t=t_0} - \left(-\frac{1}{2a}\right)\cdot\ln\left(\rho_0^{-1}-2at\right)\Big|_{t=t_1}\end{aligned} \tag{4.13}$$

for any $t_1 < t_0$ sufficiently close to t_0. Note that $\lambda_{1,p}(f(t_0),t_0) = \lambda_{1,p}(t_0)$ and $\lambda_{1,p}(f(t_1),t_1) \geq \lambda_{1,p}(t_1)$. Then (4.13) becomes

$$\ln\lambda_{1,p}(t_0) + \ln\left(\rho_0^{-1}-2at_0\right)^{\frac{1}{2a}} > \ln\lambda_{1,p}(t_1) + \ln\left(\rho_0^{-1}-2at_1\right)^{\frac{1}{2a}}.$$

Namely,

$$\lambda_{1,p}(t_0)\cdot\left(\rho_0^{-1}-2at_0\right)^{\frac{1}{2a}} > \lambda_{1,p}(t_1)\cdot\left(\rho_0^{-1}-2at_1\right)^{\frac{1}{2a}}$$

for any $t_1 < t_0$ sufficiently close to t_0. Since t_0 is arbitrary, then (4.8) follows.

Now we know that

$$\lambda_{1,p}(t)\cdot\left(\rho_0^{-1}-2at\right)^{\frac{1}{2a}}$$

is increasing along the unnormalized Ricci flow. Moreover, $\left(\rho_0^{-1}-2at\right)^{\frac{1}{2a}}$ is a smooth function. Hence by the Lebesgue's theorem, $\lambda_{1,p}(t)$ is differentiable almost everywhere along the unnormalized Ricci flow on $[0,T')$. □

Remark 4.4. Since function $\left(\rho_0^{-1}-2at\right)^{\frac{1}{2a}}$ is decreasing in t-variable, Theorem 4.3 also implies that $\lambda_{1,p}(t)$ is strictly increasing along the unnormalized Ricci flow on $[0,T')$.

We also have decreasing quantities along the unnormalized Ricci flow.

Theorem 4.5. *Let $g(t)$ and $\lambda_{1,p}(t)$ $(p>1)$ be the same as in Theorem 1.1. If*

$$0 \leq R_{ij} < \tfrac{R}{p}g_{ij}(t) \qquad \text{in} \quad M^n\times[0,T), \tag{4.14}$$

then the following quantity

$$\lambda_{1,p}(t)\cdot\left(\sigma_0^{-1}-\frac{2n}{p^2}t\right)^{\frac{p^2}{2n}} \tag{4.15}$$

is strictly decreasing and therefore $\lambda_{1,p}(t)$ is differentiable almost everywhere along the unnormalized Ricci flow on $[0,T')$, where $\sigma_0 := \sup_M R(0)$ and $T' := \min\{\frac{p^2}{2n\sigma_0}, T\}$.

Proof. The proof is similar to that of Theorem 4.3 with the difference that we need to estimate the upper bounds of the right hand side of (4.16). Here we only briefly sketch the proof. According to (4.3) of Lemma 4.1, we have

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} = \lambda_{1,p}(f(t_0),t_0)\int_M |f|^p R d\mu + \int_M |df|^{p-2}(pR_{ij}-Rg_{ij})f_if_jd\mu, \tag{4.16}$$

where f is a smooth function satisfying the same assumptions as in the proof of Theorem 4.3.

Note that $0 \leq R_{ij} < \frac{R}{p}g_{ij}$ implies $|Ric|^2 < \frac{n}{p^2}R^2$. So the evolution of the scalar curvature R under the unnormalized Ricci flow

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2$$

implies

$$\frac{\partial}{\partial t}R \leq \Delta R + \frac{2n}{p^2}R^2. \tag{4.17}$$

Applying the maximum principle to (4.17), we have

$$0 \leq R(x,t) \leq \sigma(t),$$

where

$$\sigma(t) = \frac{1}{{\sigma_0}^{-1} - \frac{2n}{p^2}t}, \quad t \in [0, T'),$$

and where $\sigma_0 := \sup_M R(0)$ and $T' := \min\{\frac{p^2}{2n\sigma_0}, T\}$.

Substituting $0 \leq R(x,t) \leq \sigma(t)$ and $0 \leq R_{ij} < \frac{R}{p}g_{ij}$ into (4.16) yields

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} < \lambda_{1,p}(f(t_0),t_0)\cdot\sigma(t_0).$$

Hence

$$\frac{d}{dt}\lambda_{1,p}(f,t) < \lambda_{1,p}(f(t),t)\cdot\sigma(t)$$

in any sufficiently small neighborhood of t_0. Integrating this inequality with respect to time t on time interval $[t_0,t_1]$ yields

$$\lambda_{1,p}(t_1)\cdot\left(\sigma_0^{-1} - \frac{2n}{p^2}t_1\right)^{\frac{p^2}{2n}} < \lambda_{1,p}(t_0)\cdot\left(\sigma_0^{-1} - \frac{2n}{p^2}t_0\right)^{\frac{p^2}{2n}}$$

for any $t_1 > t_0$ sufficiently close to t_0, where we used $\lambda_{1,p}(f,t_0) = \lambda_{1,p}(t_0)$ and $\lambda_{1,p}(f,t_1) \geq \lambda_{1,p}(t_1)$. Since t_0 is arbitrary, then Theorem 4.5 follows. □

For any closed 3-manifold, we have

Corollary 4.6. *Let $g(t)$ and $\lambda_{1,p}(t)$ be the same as in Theorem 1.1., where we assume $n=3$ and $1<p<3$. If*

$$0 \leq R_{ij}(0) < \tfrac{R(0)}{p}g_{ij}(0) \qquad \text{in} \quad M^3\times\{0\}, \tag{4.18}$$

then the conclusion of Theorem 4.5 is also true.

Remark 4.7. Note that if $p=2$, condition (4.18) is the same as positive sectional curvatures of this closed manifold.

Proof. According to Hamilton's maximum principle for tensors (see Theorem 9.6 in [11]), for $1<p<3$, we conclude that $0 \leq R_{ij} < \frac{R}{p}g_{ij}$ is preserved under the Ricci flow. Therefore the desired conclusion follows from Theorem 4.5. □

5. First p-eigenvalue along normalized Ricci flow

In this section, we will first discuss the differentiability for $\lambda_{1,p}(\tilde{g}(\tilde{t}))$ under normalized Ricci flow by means of the differentiability for $\lambda_{1,p}(g(t))$ under unnormalized Ricci flow. Then for closed 2-surfaces, we obtain many monotonic quantities about the first eigenvalue of the p-Laplace operator along the normalized Ricci flow without any curvature assumption, that is, Theorems 1.4 and 1.5 in introduction.

At first we can apply the differentiability for $\lambda_{1,p}(g(t))$ under the unnormalized Ricci flow to derive the differentiability for $\lambda_{1,p}(\tilde{g}(\tilde{t}))$ under the normalized case.

Theorem 5.1. *Let $\tilde g(\tilde t)$, $\tilde t \in [0,\infty)$, be a solution of the normalized Ricci flow (1.2) on a closed manifold M^n and let $\lambda_{1,p}(\tilde t)$ be the first eigenvalue of the p-Laplace operator of the metric $\tilde g(\tilde t)$. If the curvature assumptions of Theorem 1.1 (Theorem 4.3, Theorem 4.5 or Corollary 4.6) are satisfied, then $\lambda_{1,p}(\tilde t)$ is differentiable almost everywhere along the normalized Ricci flow on $[0,\infty)$ in each case.*

Proof of Theorem 5.1. Under the normalized Ricci flow $\tilde g(\tilde t) := c(t)g(t)$, we have

$$\lambda_{1,p}(\tilde g(\tilde t)) = \frac{\int_M |d\tilde f|^p_{\tilde g(\tilde t)} d\tilde\mu}{\int_M |\tilde f|^p d\tilde\mu} = \frac{\int_M |d\tilde f|^p_{\tilde g(\tilde t)} d\mu}{\int_M |\tilde f|^p d\mu} = c(t)^{-p/2}\frac{\int_M |d\tilde f|^p_{g(t)} d\mu}{\int_M |\tilde f|^p d\mu}, \tag{5.1}$$

where $\tilde f$ is the eigenfunction for the first eigenvalue $\lambda_{1,p}(\tilde t)$ with respect to $\tilde g(\tilde t)$, which implies $\int_M |\tilde f|^{p-2}\tilde f d\tilde\mu = 0$. Since $\tilde g(\tilde t) := c(t)g(t)$, we also have

$$\int_M |\tilde f|^{p-2}\tilde f d\mu = 0.$$

Consider the following quantity

$$\frac{\int_M |d\phi|^p_{g(t)} d\mu}{\int_M |\phi|^p d\mu}, \tag{5.2}$$

where ϕ is any C^1 function. Clearly, if $\phi = \tilde f$, then (5.2) achieves its minimum. If it is not true, this contradicts (5.1) by choosing $c(t) = 1$. Therefore (5.1) implies that

$$\lambda_{1,p}(\tilde g(\tilde t)) = c(t)^{-p/2} \cdot \lambda_{1,p}(g(t)).$$

Note that $\lambda_{1,p}(g(t))$ is differentiable almost everywhere under the curvature assumptions of Theorem 1.1 (Theorem 4.3, Theorem 4.5 or Corollary 4.6) and $c(t)$ is a smooth function. Hence $\lambda_{1,p}(\tilde t)$ is differentiable almost everywhere in each case along the normalized Ricci flow on $[0,\infty)$. □

Remark 5.2. For any 2-surface, we claim that $\lambda_{1,p}(t)$ is differentiable almost everywhere along the Ricci flow without any curvature assumption (see Theorems 1.4 and 1.5, and Corollary 5.4).

In the rest of this section, we shall discuss the monotonic quantities about the first eigenvalue of the p-Laplace operator along the normalized Ricci flow on closed 2-surfaces. From this, we also see that $\lambda_{1,p}(t)$ is differentiable almost everywhere along the normalized Ricci flow without any curvature assumption.

We recall the following curvature estimates along the normalized Ricci flow on closed surfaces (see Proposition 5.18 in [7]).

Proposition 5.3. *For any solution $(M^2, g(t))$ of the normalized Ricci flow on a closed surface, there exists a constant $C > 0$ depending only on the initial metric such that:*

(1) *If $r < 0$, then $r - Ce^{rt} \le R \le r + Ce^{rt}$.*
(2) *If $r = 0$, then $-\frac{C}{1+Ct} \le R \le C$.*
(3) *If $r > 0$, then $-Ce^{rt} \le R \le r + Ce^{rt}$.*

Now using Proposition 5.3, we shall prove Theorem 1.4. The method of proof is almost the same as that of Theorem 4.3.

Proof of Theorem 1.4. Step 1: we first prove the case $p \geq 2$. Since $n = 2$, by (4.4) of Lemma 4.1, under the normalized Ricci flow, we have

$$\begin{aligned}\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} &= \lambda_{1,p}(f(t_0),t_0)\int_M |f|^p R d\mu + \left(\frac{p}{2}-1\right)\int_M |df|^p R d\mu \\ &\quad - \frac{p}{2} r \lambda_{1,p}(f(t_0),t_0),\end{aligned} \tag{5.3}$$

where f is defined by Lemma 4.1.

Case 1: $\chi(M^2) < 0$.

Note that the evolution of the scalar curvature R on a closed surface under the normalized Ricci flow is

$$\frac{\partial}{\partial t}R = \Delta R + R(R - r). \tag{5.4}$$

By the Gauss-Bonnet theorem, r is determined by the Euler characteristic $\chi(M^2)$, i.e., $r = 4\pi\chi(M^2)/\mathrm{Area}(M^2)$. Now if $\chi(M^2) < 0$, applying the maximum principle to equation (5.4), we obtain sharp lower bounds of the scalar curvature R:

$$R(x,t) \geq \frac{r}{1-(1-\frac{r}{\rho_0})e^{rt}}, \qquad t \in [0,\infty). \tag{5.5}$$

Note that in this setting, we need more accurate lower bounds than Proposition 5.3. By inequality (5.5), we have

$$R(x,t) > \frac{r}{1-(1-\frac{r}{\rho_0})e^{rt}} - \epsilon, \qquad t \in [0,\infty) \tag{5.6}$$

for $\epsilon > 0$ sufficiently small. Substituting this into the above formula (5.3), we obtain

$$\begin{aligned}\frac{d\lambda_{1,p}(f,t)}{dt}\Big|_{t=t_0} &> \lambda_{1,p}(f(t_0),t_0)\left[\frac{r}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - \frac{p}{2}r\right] \\ &\quad + \left(\frac{p}{2}-1\right)\frac{r\lambda_{1,p}(f(t_0),t_0)}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - \frac{p\epsilon}{2}\lambda_{1,p}(f(t_0),t_0) \\ &= \frac{p}{2}\lambda_{1,p}(f(t_0),t_0)\left[\frac{r}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - r - \epsilon\right].\end{aligned} \tag{5.7}$$

Since $\lambda_{1,p}(f,t)$ is a smooth function with respect to t-variable, we have

$$\frac{d}{dt}\lambda_{1,p}(f,t) > \frac{p}{2}\lambda_{1,p}(f(t),t)\left[\frac{r}{1-(1-\frac{r}{\rho_0})e^{rt}} - r - \epsilon\right] \tag{5.8}$$

in any sufficiently small neighborhood of t_0. Integrating the above inequality with respect to time t on a sufficiently small time interval $[t_1,t_0]$, we obtain

(5.9)

$$\begin{aligned}&\ln\lambda_{1,p}(f(t_0),t_0) - \ln\lambda_{1,p}(f(t_1),t_1) \\ &\quad > \frac{p}{2}\left[\ln\frac{\frac{r}{\rho_0}e^{rt_0}}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - (r+\epsilon)t_0\right] - \frac{p}{2}\left[\ln\frac{\frac{r}{\rho_0}e^{rt_1}}{1-(1-\frac{r}{\rho_0})e^{rt_1}} - (r+\epsilon)t_1\right]\end{aligned}$$

for any $t_1 < t_0$ sufficiently close to t_0 (Note that t_1 may equal to 0). Since $\lambda_{1,p}(f(t_0), t_0) = \lambda_{1,p}(t_0)$ and $\lambda_{1,p}(f(t_1), t_1) \geq \lambda_{1,p}(t_1)$, then we have

(5.10)
$$\begin{aligned}&\ln\lambda_{1,p}(t_0) - \ln\lambda_{1,p}(t_1)\\&\quad > \frac{p}{2}\left[\ln\frac{\frac{r}{\rho_0}e^{rt_0}}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - (r+\epsilon)t_0\right] - \frac{p}{2}\left[\ln\frac{\frac{r}{\rho_0}e^{rt_1}}{1-(1-\frac{r}{\rho_0})e^{rt_1}} - (r+\epsilon)t_1\right]\end{aligned}$$

for any $t_1 < t_0$ sufficiently close to t_0. Since t_0 is arbitrary, we conclude that

$$\ln\lambda_{1,p}(t) - \frac{p}{2}\left[\ln\frac{\frac{r}{\rho_0}e^{rt}}{1-(1-\frac{r}{\rho_0})e^{rt}} - (r+\epsilon)t\right] \tag{5.11}$$

is increasing along the normalized Ricci flow. Taking $\epsilon \to 0$, we know that

$$\ln\left[\lambda_{1,p}(t)\cdot\left(\frac{\rho_0}{r} - \frac{\rho_0}{r}e^{rt} + e^{rt}\right)^{p/2}\right] \tag{5.12}$$

is non-decreasing along the normalized Ricci flow. By the Lebesgue's theorem, (5.12) is differentiable almost everywhere along the normalized Ricci flow on $[0,\infty)$. We also note that

$$\left[\frac{\rho_0}{r} - \frac{\rho_0}{r}e^{rt} + e^{rt}\right]^{p/2}$$

is a smooth function. Hence $\lambda_{1,p}(t)$ is differentiable almost everywhere along the normalized Ricci flow.

Case 2: $\chi(M^2) = 0$.

If $\chi(M^2) = 0$, i.e., $r = 0$, by Proposition 5.3, we have

$$R(x,t) \geq -\frac{C}{1+Ct}. \tag{5.13}$$

Substituting this into formula (5.3) and applying similar arguments above (in case of $\chi(M^2) \neq 0$), we can obtain the desired results.

Case 3: $\chi(M^2) > 0$.

This proof is similar to the proof of Case 2. we still use Proposition 5.3 and formula (5.3).

Step 2: we consider the case $1 < p < 2$. Since the method of proof is similar to the previous discussions, we only give some key computations.

Case 1: $\chi(M^2) < 0$.

By (5.3) and $R \leq r + Ce^{rt}$ of Proposition 5.3, we have

(5.14)
$$\begin{aligned}\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} &\geq \lambda_{1,p}(f(t_0),t_0)\left[\frac{r}{1-(1-\frac{r}{\rho_0})e^{rt_0}} + \left(\frac{p}{2}-1\right)\left(r+Ce^{rt_0}\right) - \frac{p}{2}r\right]\\&= \lambda_{1,p}(f(t_0),t_0)\left[\frac{r}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - r + \left(\frac{p}{2}-1\right)Ce^{rt_0}\right]\end{aligned}$$

where f is defined by Lemma 4.1.

Following similar arguments above, we conclude that (5.14) still holds in any sufficiently small neighborhood of t_0. Then integrating this inequality with respect

to time t on a sufficiently small time interval $[t_1, t_0]$, we obtain

$$\ln \lambda_{1,p}(f(t_0), t_0) - \ln \lambda_{1,p}(f(t_1), t_1) \geq \left[\ln \frac{\frac{r}{\rho_0}}{1-(1-\frac{r}{\rho_0})e^{rt_0}} + \left(\frac{p}{2}-1\right)\frac{C}{r}e^{rt_0} \right] - \left[\ln \frac{\frac{r}{\rho_0}}{1-(1-\frac{r}{\rho_0})e^{rt_1}} + \left(\frac{p}{2}-1\right)\frac{C}{r}e^{rt_1} \right] \tag{5.15}$$

for any $t_1 < t_0$ sufficiently close to t_0. Note that $\lambda_{1,p}(f(t_0), t_0) = \lambda_{1,p}(t_0)$ and $\lambda_{1,p}(f(t_1), t_1) \geq \lambda_{1,p}(t_1)$. Hence we have

$$\begin{aligned} &\ln \left[\lambda_{1,p}(t_0) \cdot \left(\frac{\rho_0}{r} - \frac{\rho_0}{r} e^{rt_0} + e^{rt_0} \right) \right] + \left(1 - \frac{p}{2}\right) \frac{C}{r} e^{rt_0} \\ &\geq \ln \left[\lambda_{1,p}(t_1) \cdot \left(\frac{\rho_0}{r} - \frac{\rho_0}{r} e^{rt_1} + e^{rt_1} \right) \right] + \left(1 - \frac{p}{2}\right) \frac{C}{r} e^{rt_1} \end{aligned}$$

for any $t_1 < t_0$ sufficiently close to t_0. Since t_0 is arbitrary, the result follows.

Case 2: $\chi(M^2) = 0$.

Using $-\frac{C}{1+Ct} \leq R \leq C$ of Proposition 5.3, we have

$$\begin{aligned} \frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} &= \lambda_{1,p}(f(t_0),t_0) \int_M |f|^p R d\mu + \left(\frac{p}{2}-1\right) \int_M |df|^p R d\mu \\ &\geq \lambda_{1,p}(f(t_0),t_0) \left[-\frac{C}{1+Ct_0} + \left(\frac{p}{2}-1\right) C \right] \end{aligned} \tag{5.16}$$

where f is defined by Lemma 4.1. Then using similar arguments above, we can obtain the desired results.

Case 3: $\chi(M^2) > 0$.

Using $-Ce^{rt} \leq R \leq r + Ce^{rt}$ of Proposition 5.3, we get

$$\begin{aligned} \frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} &= \lambda_{1,p}(f(t_0),t_0) \int_M |f|^p R d\mu + \left(\frac{p}{2}-1\right) \int_M |df|^p R d\mu \\ &\quad - \frac{p}{2} r \lambda_{1,p}(f(t_0),t_0) \\ &\geq \lambda_{1,p}(f(t_0),t_0) \left[-r + \left(\frac{p}{2}-2\right) Ce^{rt_0} \right] \end{aligned} \tag{5.17}$$

where f is defined by Lemma 4.1. Then using the standard discussions above, we can obtain the desired results. $\square$

In the following we will finish the proof Theorem 1.5.

Proof of Theorem 1.5. Step 1: we first prove the case $p \geq 2$.

The case $\chi(M^2) = 0$.

By Proposition 5.3, we have $R(x,t) \leq C$. Substituting this into formula (5.3),

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \frac{p}{2} \cdot C \lambda_{1,p}(f(t_0),t_0). \tag{5.18}$$

Since $\lambda_{1,p}(f,t)$ is a smooth function with respect to t-variable, we have

$$\frac{d}{dt}\lambda_{1,p}(f,t) < \frac{p}{2}\left(C+\epsilon\right)\lambda_{1,p}(f(t),t). \tag{5.19}$$

for $\epsilon > 0$ sufficiently small in any sufficiently small neighborhood of t_0. Integrating the above inequality with respect to time t on a sufficiently small time interval $[t_0, t_1]$, we get

$$\ln\lambda_{1,p}(f(t_1), t_1) - \ln\lambda_{1,p}(f(t_0), t_0) < \frac{p}{2}(C+\epsilon)\,t_1 - \frac{p}{2}(C+\epsilon)\,t_0 \tag{5.20}$$

for any $t_1 > t_0$ sufficiently close to t_0. Note that $\lambda_{1,p}(f(t_0), t_0) = \lambda_{1,p}(t_0)$ and $\lambda_{1,p}(f(t_1), t_1) \geq \lambda_{1,p}(t_1)$. So we have

$$\ln\lambda_{1,p}(t_1) - \frac{p}{2}(C+\epsilon)\,t_1 < \ln\lambda_{1,p}(t_0) - \frac{p}{2}(C+\epsilon)\,t_0$$

for any $t_1 > t_0$ sufficiently close to t_0. Since t_0 is arbitrary, taking $\epsilon \to 0$, the result follows in the case of $\chi = 0$.

The case $\chi(M^2) \neq 0$.

The method of the proof is similar to the case of $\chi(M^2) \neq 0$. Here we only give some key inequalities. Using $R \leq r + Ce^{rt}$ of Proposition 5.3 and formula (5.3), we have

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \frac{p}{2}Ce^{rt_0}\lambda_{1,p}(f(t_0), t_0) \tag{5.21}$$

where f is defined by Lemma 4.1. By similar arguments the results follows.

Step 2: we consider the case $1 < p < 2$. Similarly, we only give some key computations.

Case 1: $\chi(M^2) < 0$.

Substituting (5.5) and $R \leq r + Ce^{rt}$ of Proposition 5.3 into formula (5.3),

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \lambda_{1,p}(f(t_0), t_0)\left[\left(\frac{p}{2}-1\right)\cdot\left(\frac{r}{1-(1-\frac{r}{\rho_0})e^{rt_0}} - r\right) + Ce^{rt_0}\right] \tag{5.22}$$

where f is defined by Lemma 4.1. Then using the standard discussion as the case $\chi(M^2) = 0$, we can obtain the desired results.

Case 2: $\chi(M^2) = 0$.

Substituting $-\frac{C}{1+Ct} \leq R \leq C$ of Proposition 5.3 into formula (5.3), we have

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \lambda_{1,p}(f(t_0), t_0)\left[\left(1-\frac{p}{2}\right)\cdot\frac{C}{1+Ct_0} + C\right] \tag{5.23}$$

where f is defined by Lemma 4.1. Using similar discussion above, the result follows.

Case 3: $\chi(M^2) > 0$.

Using $-Ce^{rt} \leq R \leq r + Ce^{rt}$ of Proposition 5.3, we obtain

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \lambda_{1,p}(f(t_0), t_0)\left[\left(1-\frac{p}{2}\right)\cdot r + \left(2-\frac{p}{2}\right)Ce^{rt_0}\right] \tag{5.24}$$

where f is defined by Lemma 4.1. Then the desired results follow by the above similar discussions. □

We should point out that for closed 2-surfaces, we also have the differentiability result along the unnormalized Ricci flow without any curvature assumption.

Corollary 5.4. *Let $g(t)$ and $\lambda_{1,p}(t)$ be the same as in Theorem 1.1, where $n = 2$. Then $\lambda_{1,p}(t)$ is differentiable almost everywhere along the unnormalized Ricci flow.*

Proof. For closed 2-surfaces, we know that the first eigenvalue of the p-Laplace operator is differentiable almost everywhere along the normalized Ricci flow. Hence the conclusion follows from the same argument as in the proof of Theorem 5.1. □

6. p-EIGENVALUE COMPARISON-TYPE THEOREM

In Riemannian geometry, a convenient way of understanding a general Riemannian manifold is by comparison theorems. And many comparison theorems have been obtained, such as the Hessian comparison theorem, the Laplace comparison theorem, the volume comparison theorem, etc..

In this section, we will give another interesting comparison-type theorem on a closed surface with the Euler characteristic $\chi(M^2) < 0$, which is motivated by the work of J. Ling [18]. However, our proof may be different from Ling's. Because we do not know the eigenvalue or eigenfunction differentiability under the Ricci flow. Fortunately we can follow similar arguments above and obtain our desired result.

Let (M^2, g) be a closed surface. Let K_g, κ_g, $\mathrm{Area}_g(M^2)$ denote the Gauss curvature, the minimum of the Gauss curvature, the area of the surface, respectively. $\lambda_{1,p}(g)$ denotes the first eigenvalue of the p-Laplace operator ($p \geq 2$) with respect to the metric g. We now prove the comparison-type theorem for $\lambda_{1,p}(g)$ on a closed surface with its Euler characteristic is negative.

Proof of Theorem 1.7. Let $g(t)$ be the solution of the normalized Ricci flow on a closed surface

$$\frac{\partial g(t)}{\partial t} = (r - R)g(t) \tag{6.1}$$

with the initial condition $g(0) = g$, where R is the scalar curvature of the metric $g(t)$ and $r = \int_{M^2} R d\mu / \int_{M^2} d\mu$, which keeps the area of the surface constant. In fact, from (6.1) we have

$$\frac{d}{dt}(d\mu) = (r - R)d\mu$$

and

$$\frac{d}{dt}\mathrm{Area}_{g(t)}(M^2) = \frac{d}{dt}\int_{M^2} d\mu = \int_{M^2} (r - R)d\mu = 0.$$

Set $A := \mathrm{Area}_{g(t)}(M^2) = \mathrm{Area}_g(M^2)$. Obviously, along the normalized Ricci flow, the area A remains constant independent of time. By the Gauss-Bonnet theorem, r is determined by the Euler characteristic $\chi(M^2)$, i.e., $r = 4\pi\chi(M^2)/A < 0$. So we know that r is a negative constant and the lower bounds of the scalar curvature R are also negative. Meanwhile, according to Theorem E in introduction, the metric $g(t)$ converges to a smooth metric $\bar{g}(= g(\infty))$ of constant Gauss curvature $r/2$.

Note that $R/2$ is the Gauss curvature K of the metric $g(t)$. Let $\rho_0 < 0$ be the minimum of $R(0)$, i.e.,

$$R(0) = 2K(0) \geq \rho_0.$$

Since $\chi(M^2) < 0$, by Theorem 1.4, we know that

$$\lambda_{1,p}(t) \cdot \left[\frac{\rho_0}{r} - \frac{\rho_0}{r}e^{rt} + e^{rt}\right]^{p/2} \tag{6.2}$$

is increasing along the normalized Ricci flow on $[0, \infty)$, where $\rho_0 = \inf_{M^2} R(0)$.

Since that $r < 0$ and $p \geq 2$, taking $t \to \infty$ in (6.2) and noticing that $\lambda_{1,p}(t)$ is continuous, we conclude that

$$\lambda_{1,p}(\infty) \geq \lambda_{1,p}(0) \cdot \left(\frac{r}{\rho_0}\right)^{p/2}.$$

Note that the metric $\bar{g}(= g(\infty))$ has constant Gauss curvature $r/2$. So we have $\kappa_{\bar{g}} = r/2$. By the definition for ρ_0, we also have $\rho_0 = 2\kappa_g$. Therefore we conclude the following inequality

$$\frac{\lambda_{1,p}(\bar{g})}{\lambda_{1,p}(g)} \geq \left(\frac{\kappa_{\bar{g}}}{\kappa_g}\right)^{p/2}.$$

This completes the proof of this theorem. □

Remark 6.1. (1). By Theorem 1.4 and Theorem 1.5, using the same method above, if $\chi(M^2) < 0$, we can also get some rough estimates

$$\frac{\lambda_{1,p}(\bar{g})}{\lambda_{1,p}(g)} \geq \exp\left[\left(1 - \frac{p}{2}\right)\frac{C}{r}\right] \cdot \frac{\kappa_{\bar{g}}}{\kappa_g} \quad (1 < p < 2);$$

and

$$\frac{\lambda_{1,p}(\bar{g})}{\lambda_{1,p}(g)} \leq e^{-\frac{C}{r}} \cdot \left(\frac{\kappa_{\bar{g}}}{\kappa_g}\right)^{\frac{p}{2}-1} \quad (1 < p < 2), \qquad \frac{\lambda_{1,p}(\bar{g})}{\lambda_{1,p}(g)} \leq \exp\left(-\frac{p}{2} \cdot \frac{C}{r}\right) \quad (p \geq 2),$$

where $C > 0$ is a constant depending only on the metric g and $r = 2\kappa_{\bar{g}}$.

(2). It would be interesting to find out if there exists a similar comparison-type result for high dimensional closed manifolds. It seems to be difficult to deal with the high-dimensional case. On the other hand, can one have a similar result as theorem 1.7 if one removes the condition: $\chi(M^2) < 0$?

(3). Though we do not follow J. Ling's proof, the idea of proof partly belongs to his. When $p = 2$, our result reduces to J. Ling's (see [18], Theorem 1.1).

7. First p-eigenvalue along general evolving metrics

Following similar arguments in the proof of Theorem 1.1, in this section, we discuss the monotonicity and differentiability for the first eigenvalue of the p-Laplace with respect to general evolving Riemannian metrics.

Let $(M^n, g(t))$ be a smooth one-parameter family of compact Riemannian manifolds without boundary evolving for $t \in [0, T)$ by

$$\frac{\partial}{\partial t} g_{ij} = -2h_{ij} \tag{7.1}$$

with $g(0) = g_0$. Let $H := \operatorname{tr} h = g^{ij} h_{ij}$.

We first have a analog of Proposition 3.1 in Section 3.

Proposition 7.1. *Let $g(t)$, $t \in [0, T)$, be a smooth family of complete Riemannian metrics on a closed manifold M^n satisfying (7.1) and let $\lambda_{1,p}(t)$ be the first eigenvalue of the p-Laplace operator ($p > 1$) under the evolving metrics (7.1). For any $t_1, t_2 \in [0, T)$ with $t_2 \geq t_1$, we have*

$$\lambda_{1,p}(t_2) \geq \lambda_{1,p}(t_1) + \int_{t_1}^{t_2} \mathcal{L}(g(\xi), f(\xi)) d\xi, \tag{7.2}$$

where

$$\mathcal{L}(g(t), f(t)) := p\int_M |df|^{p-2}h(\nabla f, \nabla f)d\mu - p\int_M \Delta_p f\frac{\partial f}{\partial t}d\mu - \int_M |df|^p H d\mu \tag{7.3}$$

and where $f(t)$ is any C^∞ function satisfying the restrictions $\int_M |f(t)|^p d\mu_{g(t)} = 1$ and $\int_M |f(t)|^{p-2}f(t)d\mu_{g(t)} = 0$, such that at time t_2, $f(t_2)$ is the corresponding eigenfunction of $\lambda_{1,p}(t_2)$.

Proof. The proof is by straightforward computation, which is similar to the proof of Proposition 3.1. Here we omit those details. □

Using this proposition, we have

Theorem 7.2. *Let $g(t)$ and $\lambda_{1,p}(t)$ be the same as Proposition 7.1. If there exists a nonnegative constant ϵ such that*

$$h_{ij} - \tfrac{H}{p}g_{ij} \geq -\epsilon g_{ij} \qquad \text{in} \quad M \times [0, T) \tag{7.4}$$

and

$$H > p \cdot \epsilon \qquad \text{in} \quad M \times [0, T), \tag{7.5}$$

then $\lambda_{1,p}(t)$ is strictly increasing and therefore differentiable almost everywhere along the evolving Riemannian metrics (7.1) on $[0, T)$.

Proof. This proof is similar to that of the previous theorems. □

Remark 7.3. (1). Assumptions (7.4) and (7.5) may not be valid sometimes for some special curvature flow. For example, for the normalized Ricci flow, the assumptions (7.4) and (7.5) are not hold in general.

(2). This theorem may be compared to Theorem 1.1 of this paper. In fact, let $(M^n, g(t))$ be a complete solution of the unnormalized Ricci flow on $[0, T)$. This corresponds to $h_{ij} = R_{ij}$ and $H = R$ in Theorem 7.2.

In the following, a general version of Lemma 4.1 is stated as follows.

Lemma 7.4. *If $\lambda_{1,p}(t)$ is the first eigenvalue of $\Delta_{p_{g(t)}}$, whose metric satisfying equation (7.1) and $f(t_0)$ is the corresponding eigenfunction of $\lambda_{1,p}(t)$ at time t_0, then we have*

$$\frac{d}{dt}\lambda_{1,p}(f, t)\Big|_{t=t_0} = \lambda_{1,p}(f(t_0), t_0)\int_M |f|^p H d\mu + \int_M |df|^{p-2}(ph_{ij} - Hg_{ij})f_i f_j d\mu, \tag{7.6}$$

where $f(t)$ is any C^∞ function satisfying the restrictions $\int_M |f(t)|^p d\mu_{g(t)} = 1$ and $\int_M |f(t)|^{p-2}f(t)d\mu_{g(t)} = 0$, such that at time t_0, $f(t_0)$ is the corresponding eigenfunction of $\lambda_{1,p}(t_0)$.

In the same way as before, we can use this lemma to construct some monotonic quantities about the first eigenvalue of the p-Laplace operator along general evolving Riemannian metrics under some curvature assumptions.

Next we turn to study a particular geometric flow, i.e., Yamabe flow. We will apply Theorem 7.2 and Lemma 7.4 to the Yamabe flow. When $p = 2$, the first author

in [27] obtained some interesting results. The Yamabe flow was still introduced by R.S. Hamilton, which is defined by

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= -R(x,t)g(x,t), \\ g(x,0) &= g_0(x) \end{aligned} \tag{7.7}$$

where R denotes the scalar curvature of $g(t)$. The normalized Yamabe flow is defined by

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= (r(t) - R(x,t))\, g(x,t), \\ g(x,0) &= g_0(x) \end{aligned} \tag{7.8}$$

where $r(t) := \int_M R d\mu / \int_M d\mu$ is the average scalar curvature of the metric $g(t)$.

For the unnormalized Yamabe flow, we have the following proposition.

Proposition 7.5. *In Proposition 7.1, we replace general evolving metrics by the unnormalized Yamabe flow (7.7). Then for any $t_1, t_2 \in [0,T)$ with $t_2 \geq t_1$,*

$$\lambda_{1,p}(t_2) \geq \lambda_{1,p}(t_1) + \int_{t_1}^{t_2} \mathcal{L}(g(\xi), f(\xi)) d\xi, \tag{7.9}$$

where

$$\mathcal{L}(g(t), f(t)) := \frac{p-n}{2} \int_M |df|^p R d\mu - p \int_M \Delta_p f \frac{\partial f}{\partial t} d\mu. \tag{7.10}$$

Proof. Substituting $h_{ij} = \frac{R}{2} g_{ij}$ into Proposition 7.1, the result follows. □

Using this proposition, we have

Theorem 7.6. *Let $g(t)$ and $\lambda_{1,p}(t)$ be the same as Proposition 7.5, where we assume $p \geq n$. If*

$$R \geq 0 \quad \text{and} \quad R \not\equiv 0 \qquad \text{in} \quad M^n \times \{0\}, \tag{7.11}$$

then $\lambda_{1,p}(t)$ is strictly increasing and therefore differentiable almost everywhere along the unnormalized Yamabe flow (7.7) on $[0,T)$.

Proof of Theorem 7.6. Using basically the same trick as in proving Theorem 1.1, we shall prove this result. Under the Yamabe flow (7.7), from the constraint condition

$$\frac{d}{dt} \int_M |f(t)|^p \, d\mu_{g(t)} = 0,$$

we have

$$p \int_M |f|^{p-2} f \frac{\partial f}{\partial t} d\mu = \frac{n}{2} \int_M |f|^p R d\mu. \tag{7.12}$$

Note that at time t_2, $f(t_2)$ is the eigenfunction for the first eigenvalue $\lambda_{1,p}(t_2)$ of $\Delta_{p_{g(t_2)}}$. Therefore at time t_2, we have

$$\Delta_p f(t_2) = -\lambda_{1,p}(t_2) |f(t_2)|^{p-2} f(t_2). \tag{7.13}$$

By Proposition 7.5, at time t_2, we have

$$\begin{aligned}\mathcal{L}(g(t_2), f(t_2)) &= \frac{p-n}{2}\int_M |df|^p R d\mu - p\int_M \Delta_p f \frac{\partial f}{\partial t} d\mu \\ &= \frac{p-n}{2}\int_M |df|^p R d\mu + p\lambda_{1,p}(t_2)\int_M |f|^{p-2} f \frac{\partial f}{\partial t} d\mu \\ &= \frac{p-n}{2}\int_M |df|^p R d\mu + \frac{n}{2}\lambda_{1,p}(t_2)\int_M |f|^p R d\mu,\end{aligned} \tag{7.14}$$

where we used (7.13) and (7.12). Notice that the evolution of the scalar curvature R under the Yamabe flow (7.7) (see [6]) is

$$\frac{\partial}{\partial t} R = (n-1)\Delta R + R^2. \tag{7.15}$$

Applying the strong maximum principle, $R(g(0)) \geq 0$ and $R(x_0, 0) > 0$ for some $x_0 \in M^n$ imply that $R(x,t) > 0$ for all $(x,t) \in M^2 \times (0,T)$. Since $p \geq n$, from (7.14), we then have $\mathcal{L}(g(t_2), f(t_2)) > 0$. Then using the same arguments in proving Theorem 1.1 yields the desired result. □

For the normalized Yamabe flow, we have

Lemma 7.7. *If $\lambda_{1,p}(t)$ is the first eigenvalue of $\Delta_{p_{g(t)}}$, whose metric satisfying normalized Yamabe flow (7.8) and $f(t_0)$ is the corresponding eigenfunction of $\lambda_{1,p}(t)$ at time t_0, then we have*

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} = \frac{p-n}{2}\int_M |df|^p (R-r) d\mu + \frac{n}{2}\lambda_{1,p}(f(t_0), t_0)\int_M |f|^p (R-r) d\mu, \tag{7.16}$$

Proof. Substituting $h_{ij} = \frac{R-r}{2} g_{ij}$ into Lemma 7.4, then the result follows. □

In the end of this section, we will apply Lemma 7.7 to construct some monotonic quantities along the unnormalized Yamabe flow, generalizing earlier results for $p = 2$ derived by the first author in [27].

Theorem 7.8. *Let $g(t)$, $t \in [0,T)$, be a solution of the unnormalized Yamabe flow (7.7) on a closed manifold M^n and let $\lambda_{1,p}(t)$ be the first eigenvalue of the p-Laplace operator of the metric $g(t)$. Assume that the initial scalar curvature $R(g(0)) > 0$. Then on one hand, if $1 < p < n$,*

$$\lambda_{1,p}(t)\cdot(1-\rho_0 t)^{n/2}\cdot(1-\sigma_0 t)^{\frac{p-n}{2}} \tag{7.17}$$

is increasing along the unnormalized Yamabe flow on $[0, T'')$ and if $p \geq n$,

$$\lambda_{1,p}(t)\cdot(1-\rho_0 t)^{p/2} \tag{7.18}$$

is increasing along the unnormalized Yamabe flow on $[0, T')$. On the other hand, the following quantities

$$\lambda_{1,p}(t)\cdot(1-\rho_0 t)^{\frac{p-n}{2}}\cdot(1-\sigma_0 t)^{n/2} \qquad (1 < p < n) \tag{7.19}$$

and

$$\lambda_{1,p}(t)\cdot(1-\sigma_0 t)^{p/2} \qquad (p \geq n) \tag{7.20}$$

are both decreasing along the unnormalized Yamabe flow on $[0, T'')$, where $\rho_0 := \inf_{M^2} R(0)$, $\sigma_0 := \sup_{M^2} R(0)$, $T' := \min\{\rho_0^{-1}, T\}$ and $T'' := \min\{\sigma_0^{-1}, T\}$. Therefore $\lambda_{1,p}(t)$ is differentiable almost everywhere along the unnormalized Yamabe flow.

Proof. Since this proof is similar to the proofs of Theorems 1.4 and 1.5, we only give some key inequalities. Note that under the unnormalized Yamabe flow,

$$\frac{\partial}{\partial t}R = (n-1)\Delta R + R^2.$$

Applying the maximum principle to this equation, we have lower and upper bounds of the scalar curvature R

$$R(x,t) \geq \frac{\rho_0}{1-\rho_0 t}, \quad t \in [0,T'); \qquad R(x,t) \leq \frac{\sigma_0}{1-\sigma_0 t}, \quad t \in [0,T''). \tag{7.21}$$

where $\rho_0 := \inf_{M^n} R(0)$, $\sigma_0 := \sup_{M^n} R(0)$, $T' := \min\{\rho_0^{-1}, T\}$ and $T'' := \min\{\sigma_0^{-1}, T\}$.

By (7.16) of Lemma 7.7, we also have

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} = \frac{p-n}{2}\int_M |df|^p R d\mu + \frac{n}{2}\lambda_{1,p}(f(t_0),t_0)\int_M |f|^p R d\mu, \tag{7.22}$$

where f is defined by Lemma 7.7.

On one hand, if $1<p<n$, by (7.21) and (7.22) we conclude

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \geq \lambda_{1,p}(f(t_0),t_0)\left[\frac{p-n}{2}\cdot\frac{\sigma_0}{1-\sigma_0 t_0} + \frac{n}{2}\cdot\frac{\rho_0}{1-\rho_0 t_0}\right].$$

Then following the exactly same arguments as in proving Theorem 1.4, we see that

$$\lambda_{1,p}(t)\cdot(1-\rho_0 t)^{n/2}\cdot(1-\sigma_0 t)^{\frac{p-n}{2}}$$

is increasing along the unnormalized Yamabe flow on $[0,T'')$.

If $p \geq n$, by (7.21) and (7.22) we have

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \geq \frac{p}{2}\lambda_{1,p}(f(t_0),t_0)\cdot\frac{\rho_0}{1-\rho_0 t_0}.$$

Then using our standard arguments, we conclude that

$$\lambda_{1,p}(t)\cdot(1-\rho_0 t)^{p/2}$$

is increasing along the unnormalized Yamabe flow on $[0,T')$.

On the other hand, we consider the decreasing quantities under the unnormalized Yamabe flow. If $1<p<n$, by (7.21) and (7.22), we can get

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \lambda_{1,p}(f(t_0),t_0)\left[\frac{p-n}{2}\cdot\frac{\rho_0}{1-\rho_0 t_0} + \frac{n}{2}\cdot\frac{\sigma_0}{1-\sigma_0 t_0}\right].$$

Using the same arguments as in proving Theorem 1.5, then (7.19) follows.

If $p \geq n$, by (7.21) and (7.22), we can obatin

$$\frac{d}{dt}\lambda_{1,p}(f,t)\Big|_{t=t_0} \leq \frac{p}{2}\lambda_{1,p}(f(t_0),t_0)\cdot\frac{\sigma_0}{1-\sigma_0 t_0}.$$

By the standard arguments of Theorem 1.5, we conclude that

$$\lambda_{1,p}(t)\cdot(1-\sigma_0 t)^{p/2}$$

is decreasing along the unnormalized Yamabe flow on $[0,T'')$. □

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